Iterated Differentiable Maps with Nowhere Differentiable Basin Boundaries

Wolfgang Metzler

Fachbereich Mathematik/Informatik, Universität Kassel (GhK), W-3500 Kassel, FRG

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The fractal basin boundary of a two-dimensional discrete dynamical system modelling a chaotic forcing applied to bistability is shown to be identical to the graph of an infinite series $F(x,t) = \sum_{k=0}^{\infty} t^k f^{\circ k}(x)$ of weighted iterates of an ergodic unimodal interval function f. In the special case, when f is the logistic map in "full chaos", i.e. $f\colon x \mapsto 4x(1-x)$, F is a nowhere differentiable function of x for each $t > \exp(-\lambda^f)$ (even equal to the Weierstrass function), where $\lambda^f > 0$ is denoting the Lyapunov exponent of f. For further chaotic functions f, nowhere-differentiability is shown to be obvious from computer simulations.

1. Introduction

The study of continuous nowhere differentiable functions goes back to Weierstrass who proved in 1872 (cf. [1, 2]) that the continuous real function

$$w\colon z \mapsto \sum_{k=0}^{\infty} a^k \cos(b^k \pi z), \tag{1}$$

where b is an odd integer and $0 < a < 1$, $ab > 1 + \frac{3}{2}\pi$, has no finite or infinite differential quotient at any point $z \in \mathbb{R}$. It is well known that this result has been improved in 1916 by Hardy [3] who has shown that (1) is nowhere differentiable in $\mathbb{R}$ if $0 < a < 1$, $b > 1$ and $ab \geq 1$. Today, using methods of functional analysis, we even know that the set of functions in $C_{\mathbb{R}}(\mathbb{R})$ (denoting the set of all bounded continuous real-valued functions on $\mathbb{R}$ provided with the uniform metric) that are differentiable at at least one point of $\mathbb{R}$ is of second category in $C_{\mathbb{R}}(\mathbb{R})$. Thus, functions that are nowhere differentiable are considerably abundant in $C_{\mathbb{R}}(\mathbb{R})$. Indeed, they are dense in $C_{\mathbb{R}}(\mathbb{R})$ at least. This result is a consequence of Baire's category theorem (cf. Brown and Page [4], p. 305).

Fractal structures arise in basin boundaries of analytical maps in the complex plane (i.e. Julia sets and the boundaries of Mandelbrot sets), but also in the nonanalytical case (cf. [5] and refs. therein). Since 1990, some papers even conjecture nowhere-differentiability for boundaries of simple generic two-dimensional maps (cf. [6, 7, 8] and others). However, their assertions solely lean on experimental results.

Reprint requests to Dr. W. Metzler, Fachbereich Mathematik/Informatik, Universität Kassel (GhK), Heinrich-Plett-Str. 40, W-3500 Kassel.

2. A Conjecture on Nowhere-Differentiability

Recently, Peinke et al. [6] proposed a class ot two-dimensional discrete dynamical model systems initiated by the idea that fractal basin boundaries of generic dynamical systems are due to a chaotic forcing applied to bistability (cf. [7, 8]). The map involved reads in general

$$x_{n+1} = f(x_n),$$
$$y_{n+1} = g(y_n) + b x_n \quad (0 < b < 1), \tag{2}$$

where $f\colon [0,1] \to [0,1]$ is assumed to be C^1, unimodal and chaotic, and further $g\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is supposed to be $C^1(\mathbb{R})$, strictly increasing, bistable with two sinks, 0 and ∞, and one repelling fixed point at 1. Simple examples are given by $f\colon x \mapsto 4x(1-x)$ and $g\colon y \mapsto y^3$, respectively.

Obviously, the basin of divergence is given by

$$D = \left\{(x_0, y_0) \in [0,1] \times \mathbb{R}_0^+ \mid x_n^2 + y_n^2 \xrightarrow[n\to\infty]{} \infty\right\}, \tag{3}$$

and an easy criterion for divergence is

$$(x_0, y_0) \in D \quad \Leftrightarrow \quad \exists n \in \mathbb{N}\colon\ y_n > 1\,.$$

We are interested in the boundary between convergence and divergence in the square $[0,1] \times [0,1]$ of the first quadrant.

The boundary of D is given by

$$\left\{(x, Z(x)) \in [0,1] \times \mathbb{R}_0^+ \mid Z(x) = \lim_{n\to\infty} Z_n(x)\right\}, \tag{4}$$

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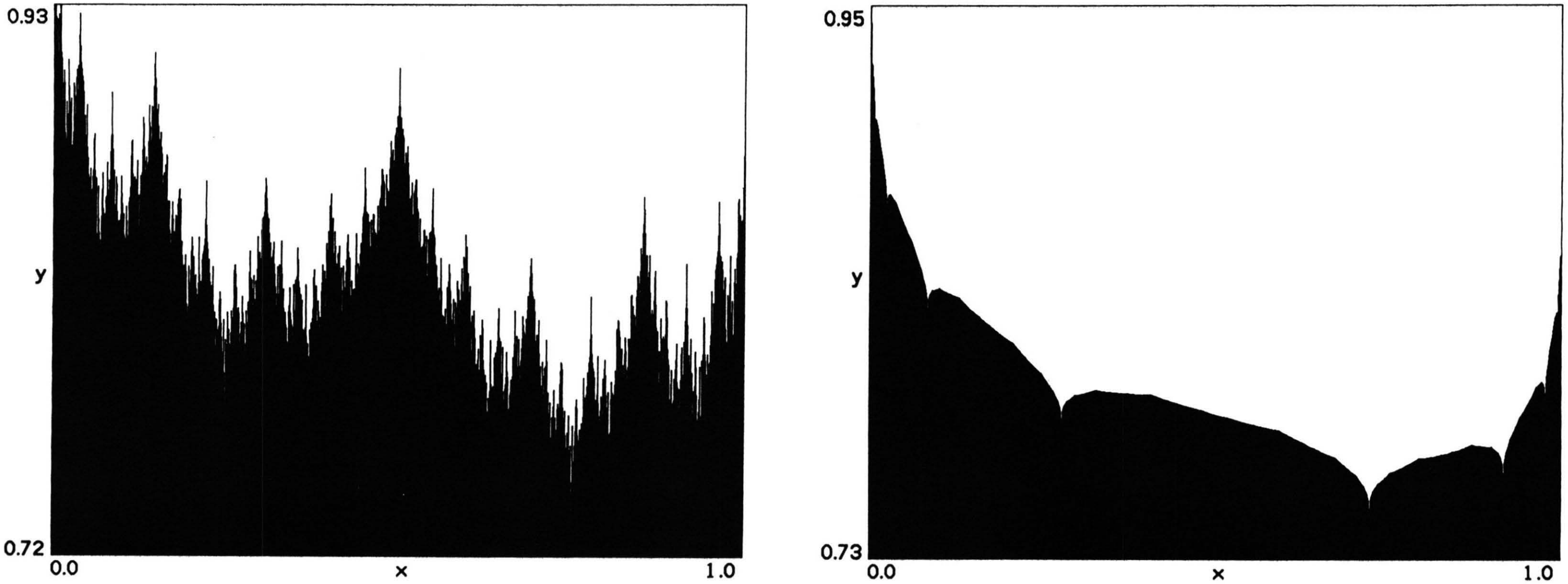

Fig. 1. Boundary function Z (complement of D colored black) from (4) for $f\colon\ x \mapsto r\,x(1-x)$ and $g\colon\ y \mapsto y^{\alpha}$ with
a) $b = 0.1,\quad r = 4,\quad \alpha = 1.352,\quad \lambda^f - \lambda^g > 0$; b) $b = 0.1,\quad r = 3.6,\quad \alpha = 1.352,\quad \lambda^f - \lambda^g < 0$.

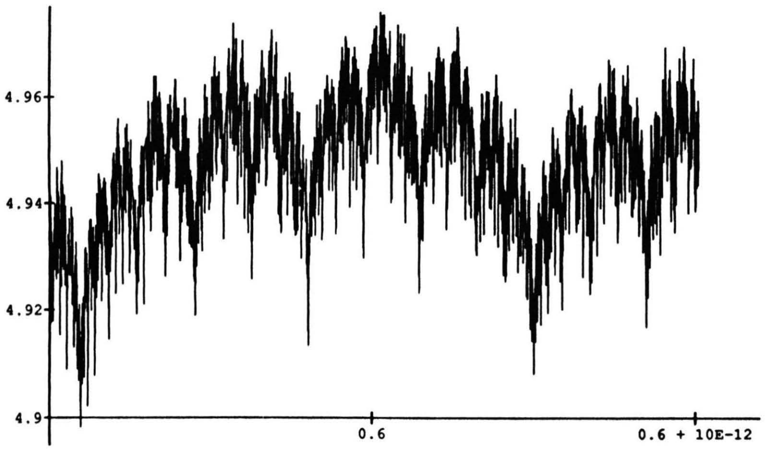

Fig. 2. Weierstrass function F from (9) for $f\colon\ x \mapsto 4\,x(1-x)$ and t = 0.9.

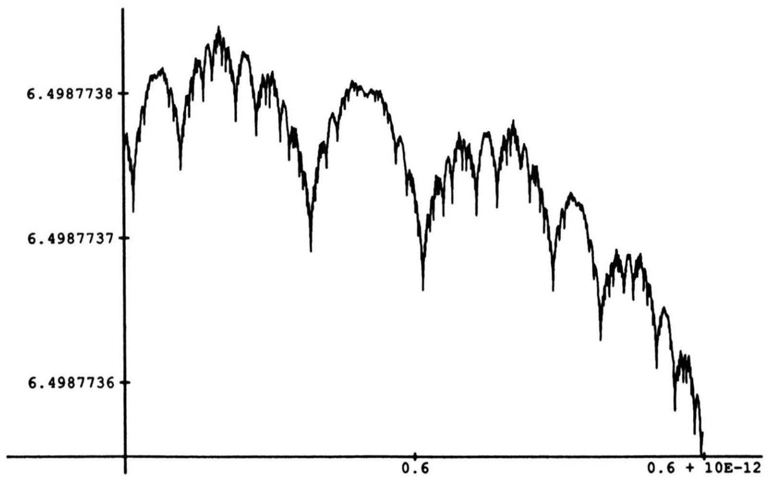

Fig. 3. Weierstrass function F from (9) for $f\colon\ x \mapsto 3.6\,x(1-x)$ and $t = 0.9$.

where $Z_n(x)$ can be defined by recursion, i.e.

$$Z_1(x) = g^{-1}(1 - bx),$$
$$Z_{n+1}(x) = g^{-1}(Z_n(f(x)) - bx), \quad (5)$$

$x \in [0, 1]$, $n \in \mathbb{N}$. Explicitly, Z_n is given by the formula (cf. [6])

$$Z_n(x) = \underbrace{g^{-1}(g^{-1}(\dots g^{-1}(g^{-1}}_{n \text{ times}}(1 - bf^{\circ(n-1)}(x)) - bf^{\circ(n-2)}(x)) \dots - bf(x)) - bx). \quad (6)$$

Model assumptions yield for each $n \in \mathbb{N}$ and $x \in [0, 1]$:

(a) $Z_n(0) = 1, \quad Z_n(1) = g^{-1}(1 - b) \in (0, 1)$,

(b) $Z_{n+1}(x) \le Z_n(x) \le 1$.

The conjecture in question on Z reads, roughly spoken (cf. [6, 8, 9]):

Z is nowhere differentiable on $[0, 1]$. (C1)

Numerous numerical simulations have shown experimental evidence to this conjecture (see [6, 8, 9] and Figure 1a). But up to now, any proof is missing.

Defining time means

$$\lambda_n^f(x) = \frac{1}{n} \sum_{k=1}^{n} \ln |f'(f^{\circ(k-1)}(x))| \quad (n \in \mathbb{N}),$$
$$\lambda^f(x) = \lim_{n \to \infty} \lambda_n^f(x) \quad (7)$$

and $\lambda_n^g(x)$, $\lambda^g(x)$ analogously (with g instead of f), conjecture (C1) can be reformulated more precise:

$\lambda^f - \lambda^g > 0 \quad \Rightarrow \quad$ *Z is nowhere differentiable (and* graph *Z is a fractal curve).*

The authors of [6] additionally assert that Z is differentiable for $\lambda^f - \lambda^g < 0$. They give no proof, but this assertion also seems to be evident from experiments (see Fig. 1b and refs. from above). Since f is assumed to be ergodic, λ^f exactly is its Lyapunov (characteristic) exponent and therefore almost everywhere independent of x. λ^g can be interpreted as rate of exponential escape from the boundary graph Z, but by no means as a characteristic exponent. Nevertheless, experiments indicate that λ^g should be invariant on $[0, 1]$ as well as λ^f.

In an earlier context [10], different from the present problem, conjecture (C1) has already been formulated by Okniński. Defining

$$F_n(x, t) = \sum_{k=0}^{n} t^k f^{\circ k}(x) \quad (0 < t < 1), \quad (8)$$

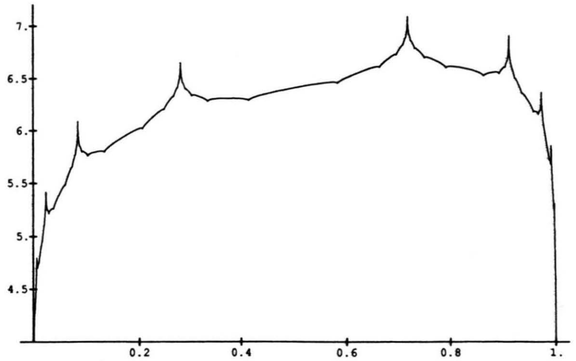

Fig. 4. Weierstrass function F from (9) for $f: x \mapsto 3.56x(1-x)$ and $t = 0.9$ (stable 8-cycle).

where $f: I \to I$ ($I \subset \mathbb{R}$ interval) is unimodal and possesses sensitive dependence on initial conditions (i.e. $\lambda^f > 0$ on I), and

$$F(x, t) = \lim_{n \to \infty} F_n(x, t) = \sum_{k=0}^{\infty} t^k f^{\circ k}(x), \quad (9)$$

we conjecture [11], that for each $t > \exp(-\lambda^f)$ the transform F is nowhere differentiable (with respect to x) at least on an open subset $X \subset I$ with positive Lebesgue measure (C2). Okniński [10] conjectured nowhere-differentiability on $X = \mathring{I} = (0, 1)$ (but gave an incorrect proof).

A connection between the present boundary map Z and Okniński's transform F can be established with the help of the derivative of Z_n. Some lengthy but straighforward transformations give

$$Z_n'(x) = \sum_{k=1}^{n-1} \sigma_k(x)\, t^k \exp(k \Delta_k(x)), \quad (10)$$

where $0 < t < 1$ and

$$\sigma_k(x) = \operatorname{sgn}\left(\frac{\mathrm{d}}{\mathrm{d}x} f^{\circ k}(x)\right),$$
$$\Delta_k(x) = \lambda_k^f(x) - \lambda_k^g(x).$$

On the other hand, we get from (7)

$$F_n'(x, t) = \sum_{k=0}^{n} t^k \frac{\mathrm{d}}{\mathrm{d}x} f^{\circ k}(x) = \sum_{k=0}^{n} \sigma_k(x)\, t^k \exp(k \lambda_k^f(x)). \quad (11)$$

Obviously, (10) and (11) are identical up to the (finite) summation range. Thus, respecting the condition on t, the conjectures (C1) and (C2) are equivalent.

For a special case, where f is the logistic map

$$f\colon\ x \mapsto r x(1-x) \tag{12}$$

in "full chaos" on its domain [0, 1], i.e. $r=4$, conjecture (C2) can be proved, cf. [12]. In this case, for $t > \exp(-\lambda^f) = \frac{1}{2}$, F is equal to the Weierstrass function up to a constant (cf. (1) and Fig. 2)

$$w\colon\ z \mapsto \sum_{k=0}^{\infty} t^k \cos(2^k \pi z)\,, \tag{13}$$

where $\pi z = \arccos(1-2x)$. But, of course, for values of $r < 4$ in the chaotic range [13], graph F is a fractal curve, too (Fig. 3), and it is (piecewise) smooth else (Fig. 4).

3. About the Converse Problem

The converse question, i.e.

f *is nowhere differentiable* $\Rightarrow d_H(\text{graph } f) > 1$,

where d_H denotes the Hausdorff-dimension of graph f, has been posed for the first time in 1977 by Mandelbrot [14] for the so-called Mandelbrot-Weierstrass function

$$m\colon\ t \mapsto \sum_{k=-\infty}^{\infty} \frac{1-\cos(b^k t)}{b^{(2-\delta)k}} \quad (1 < \delta < 2, b > 1)\,. \tag{14}$$

Tél [15] delivers a scaling relation

$$m(b t) = b^{2-\delta}\, m(t) \tag{15}$$

(which follows from (14) by a formal replacement of n by $n+1$) according to which the graph of m on the interval $[t_0, b t_0]$, to arbitrary, can be obtained by magnifying graph m in the range $[t_0/b, t_0]$ with factors b in horizontal and $b^{2-\delta}$ in vertical direction, respectively. This nontrivial symmetry is called self-affinity of graph m (cf. [15]).

Finally, already in 1980 Berry and Lewis [16] have shown analytically that

$$d_H(\text{graph } m) = \delta > 1 \tag{16}$$

in the parameter range of (14).

4. Outlook

Any (correct) proof of any conjecture on nowhere-differentiability of a fractal basin boundary gives support to an embedding of fractal structures into analytical mathematics without using tools or methods dealing with their fractal (Hausdorff) dimensions. Therefore, this paper together with [11] and [12] demonstrate a way how to be successful in that direction. Moreover, this method even seems to be generalizable.

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